ELSEVIER

Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

Discrete Mathematics 274 (2004) 281–287

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

Tree-width, clique-minors, and eigenvalues☆

Yuan Hong

Department of Mathematics, East China Normal University, Shanghai 200062, People's Republic of China

Received 15 January 2002; received in revised form 20 March 2003; accepted 2 April 2003

Abstract

Let G be a simple graph with n vertices and $\mathrm{tw}(G)$ be the *tree-width* of G. Let $\rho(G)$ be the spectral radius of G and $\lambda(G)$ be the smallest eigenvalue of G. The join $G\nabla H$ of disjoint graphs of G and H is the graph obtained from $G+H$ by joining each vertex of G to each vertex of H. In this paper, several results which are concerned with *tree-width*, *clique-minors*, and *eigenvalues* of graphs are given. In particular, we have

(1) If G is K_5 *minor-free* graph, then

$$\rho(G) \leqslant 1 + \sqrt{3n-8},$$

where equality holds if and only if G is isomorphic to $K_3\nabla(n-3)K_1$.

(2) If G is K_5 *minor-free* graph with $n \geqslant 5$ vertices, then

$$\lambda(G) \geqslant -\sqrt{3n-9},$$

where equality holds if and only if G is isomorphic to $K_{3,n-3}$.

Keywords: Graph minor; Tree-width; Surface; Eigenvalue

1. Introduction

First, we recall some basic notations that will be used in the paper.

In this paper, all graphs are finite undirected graphs without loops and multiple edges. Let $\delta(G)$ be the minimum degree of the vertices of G. The spectral radius $\rho(G)$ of G is the largest eigenvalue of its adjacency matrix. Let $\lambda(G)$ be the smallest

☆ Research supported by the Natural Science Foundation of China No. 19971027.

E-mail address: yhong@math.ecnu.edu.cn (Y. Hong).

0012-365X/03/$ - see front matter © 2003 Elsevier B.V. All rights reserved.
doi:10.1016/S0012-365X(03)00199-7

eigenvalue of G. The join $G\nabla H$ of disjoint graphs of G and H is the graph obtained from $G+H$ by joining each vertex of G to each vertex of H. A graph H is a minor of G if H can be obtained from G by deleting edges, contracting edges, and deleting isolated vertices. A graph G is *H–minor-free* if H is not a minor of G. Also, given a set S of graphs, a graph G is *S–minor-free* if G is *H–minor-free* for every $H\in S$.

An important measurement of the complexity of graphs is the concept of tree-width. Many NP-hard problems, for example, finding a maximum independent set, computing the chromatic number, and Hamiltonian cycle can be solved in linear time for graphs of bounded tree-width. There are several equivalent definitions of tree-width. The following is due to Arnborg and Proskurowski [1] (or see [4]).

Let a *k-clique* be a clique on k vertices. For a nonnegative integer k, a *k-tree* is defined inductively as follows: a *k-clique* (the complete graph K_k) is a *k-tree*. Any graph obtained from a *k-tree* by adding a new vertex and joining it to all the vertices of some *k-clique* of G is a *k-tree*. A partial *k-tree* is a subgraph of a *k-tree*. The *tree-width* of a graph G, in symbols $\mathrm{tw}(G)$, is the minimum integer k such that G is a partial *k-tree*. The terminology not defined here can be found in [2] or [3].

In this paper, the following results are obtained:

(1) If G is K_5 *minor-free* graph with n vertices, then

$$\rho(G)\leqslant 1+\sqrt{3n-8},$$

where equality holds if and only if G is isomorphic to $K_3\nabla(n-3)K_1$.

(2) If G is K_5 *minor-free* graph with $n\geqslant 5$ vertices, then

$$\lambda(G)\geqslant -\sqrt{3n-9},$$

where equality holds if and only if G is isomorphic to $K_{3,n-3}$.

(3) If $\mathrm{tw}(G)=k$, then

$$\rho(G)\leqslant \frac{k-1+\sqrt{4kn-(k+1)(3k-1)}}{2},$$

where equality holds if and only if G is isomorphic to $K_k\nabla(n-k)K_1$.

(4) If $\mathrm{tw}(G)=k$ and $n\geqslant 2k-1$, then

$$\lambda(G)\geqslant -\sqrt{k(n-k)},$$

where equality holds if and only if G is isomorphic to $K_{k,n-k}$.

2. Tree-width

It is easy to show that the following lemma holds:

Lemma 2.1. *Let G be a k-tree with n ($\geqslant k+1$) vertices and m edges. Then*

$$m=kn-\frac{k(k+1)}{2}$$

and

$$\delta(G)=k.$$

Proof. We prove this result by induction on n. It is true when $n = k + 1$ since G is the complete graph K_{k+1}. Suppose it is true when $n = N - 1$. Let G be a k-*tree* with N vertices, it is easy to see that $\delta(G) = k$. Let v be a vertex of G and $d_G(v) = \delta$, and let $H \cong G - v$, then H is a k-*tree* with $N - 1$ vertices. By the induction hypothesis, we have

$$m = K(N-1) - \frac{k(k+1)}{2} + k = kN - \frac{k(k+1)}{2} \quad \text{and} \quad \delta(G) = k. \qquad \square$$

Lemma 2.2. *Let G be a simple graph with n vertices and m edges. Let $\delta = \delta(G)$ be the minimum degree of vertices of G and $\rho(G)$ be the spectral radius of the adjacency matrix A of G. If $\delta(G) \geqslant k$, then*

$$\rho(G) \leqslant \frac{k - 1 + \sqrt{(k+1)^2 + 4(2m - kn)}}{2}.$$

Here equality holds if and only if $\delta = k$ and G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n - 1$.

Proof. This follows from Theorem 2.3 and Proposition 2.2 in [6]. $\square$

Theorem 2.1. *Let G be a simple graph with n vertices and $\rho(G)$ be the spectral radius of G. If* $\mathrm{tw}(G) = k$, *then*

$$\rho(G) \leqslant \frac{k - 1 + \sqrt{4kn - (k+1)(3k-1)}}{2}.$$

Equality holds if and only if G is isomorphic to $K_k \nabla (n-k) K_1$.

Proof. If $n = k$, then $\rho(G) \leqslant k - 1$. It is easy to see that the theorem follows. We may assume $n \geqslant k + 1$. In this case, Theorem 2.1 follows from Lemmas 2.1 and 2.2, since the graph $K_k \nabla (n-k) K_1$ is a simple graph and $\mathrm{tw}(K_k \nabla (n-k) K_1) = k$. $\square$

Corollary 2.1. *Let G be a tree with n vertices, then*

$$\rho(G) \leqslant \sqrt{n-1},$$

where equality holds if and only if G is isomorphic to $K_{1,n-1}$.

Corollary 2.2. *Let G be a series–paralled graph with n vertices, then*

$$\rho(G) \leqslant \frac{1 + \sqrt{8n - 15}}{2},$$

where equality holds if and only if G is isomorphic to $K_2 \nabla (n-2) K_1$.

Lemma 2.3. *Let G be a simple bipartite graph with n vertices and m edges. If* $\mathrm{tw}(G) \leqslant k$ *and $n \geqslant 2k - 1$, then*

$$m \leqslant k(n-k).$$

Proof. We prove this result by induction on n. If $n=2k-1$, then the complete bipartite graph $K_{k,k-1}$ is a subgraph of a k-tree $K_k\nabla(n-k)K_1$. The lemma follows. We may assume that $n>2k-1$. Suppose it is true when $n=N-1$. Let G be a k-tree with N vertices. Let v be a vertex of G and G_v denotes the graph obtained from G by deleting the vertex v together with the edges incident to v. Then G_v is a k-tree and by the induction hypothesis, the edge number of the complete bipartite subgraphs of G_v is at most $k(N-1-k)$. Hence, the edge number of the complete bipartite subgraphs of G is at most $k(N-k)$. □

Lemma 2.4. *If G is a simple connected graph with n vertices, then there exists a connected bipartite subgraph H of G such that*

$$\lambda(G)\geqslant\lambda(H)$$

with equality holding if and only if $G\cong H$.

Proof. See [5]. □

Lemma 2.5. *If G is a connected bipartite with n vertices and m edges, then*

$$\lambda(G)\geqslant-\sqrt{m},$$

where equality holds if and only if G is a complete bipartite graph.

Proof. See [5]. □

Theorem 2.2. *Let G be a simple connected graph with $n\geqslant 3$ vertices and tree-width* $\mathrm{tw}(G)=k$. *If $n\geqslant 2k-1$, then*

$$\lambda(G)\geqslant-\sqrt{k(n-k)},$$

where equality holds if and only if G is isomorphic to the complete bipartite graph $K_{k,n-k}$.

Proof. Let m be the edge number of G. By Lemmas 2.3–2.5, we get

$$\lambda(G)\geqslant-\sqrt{m}\geqslant-\sqrt{k(n-k)},$$

here equality holds if and only if G is isomorphic to the complete bipartite graph $K_{k,n-k}$. □

3. Clique-minors

The intersection $G\cap H$ of G and H is the graph with vertex set $V(G)\cap V(H)$ and edge set $E(G)\cap E(H)$. The union $G\cup H$ of G and H is defined similarly.

Suppose G is a connected graph and S be a minimal separating vertex set of G. Then we can write $G=G_1\cup G_2$, where G_1 and G_2 are connected and $G_1\cap G_2=G(S)$.

Now suppose further that $G(S)$ is a complete graph. We say that G is a *k-sum* of G_1 and G_2, denoted by $G \equiv G_1 \oplus G_2$, if $|S| = k$. Moreover, if G_1 or G_2 (say G_1) has a separating vertex set which induces a complete graph, then we can write $G_1 = G_3 \cup G_4$ such that G_3 and G_4 are connected and $G_3 \cap G_4$ is a complete subgraph of G. We proceed like this until none of the resulting subgraphs $G_1, G_2, \ldots, G_t$ has a complete separating subgraph. The graphs $G_1, G_2, \ldots, G_k$ are called the simplical summands of G. It is easy to show that the subgraphs $G_1, G_2, \ldots, G_t$ are independent of the order in which the decomposition is carried out (see [7]). Further, we denote it by $G \equiv \oplus_{i=1}^{t} G_i$.

For $n \geqslant 2$, let V_{2n} be the graph with vertex set $v_1, \ldots, v_{2n}$ and edge set $v_1v_2, \ldots, v_{2n-1}v_{2n}$, $v_1v_{n+1}, \ldots, v_nv_{2n}$.

In 1937, Wagner formulated a fundamental characterization of the graphs having no K_5 as a minor as follows:

Theorem 3.1 (Wagner equivalence theorem Thomassen [7], Wagner [8]). *Let G be an edge-maximal K_5* minor-free *graph having only one simplicial summand* (*i.e. G has no separating complete graph*), *then G is either a maximal planar graph or isomorphic to the graph V_8.*

Using Wagner Theorem, it is easy to show

Lemma 3.1. *The edge-maximal k_5* minor-free *graphs are obtained by successively pasting maximal planar graphs and copies of V_8 together along complete subgraphs. Conversely*, *if we form a graph of this type by graphs we paste together*, *then the resulting graph is an edge-maximal K_5* minor-free *graph.*

Proof. See [7]. □

Proposition 3.1. *Let G be an edge maximal K_5* minor-free *graph with n vertices and m edges*, *then*

$$m \leqslant 3n - 6.$$

Proof. Suppose that the graphs $G_1, G_2, \ldots, G_t$ are the simplical summands of G. We prove this result by induction on t. If $t = 2$, let G_i with m_i edges and n_i vertices ($i = 1, 2$), be either an edge-maximal planar graph or the graph V_8. It is easy to see that $m_i \leqslant 3n_i - 6$. If G is a p-summing of G_1 and G_2, then

$$n = n_1 + n_2 - p$$

and

$$m \leqslant 3n_1 - 6 + 3n_2 - 6 - \frac{p(p-1)}{2}.$$

Therefore, we have

$$m \leqslant 3n - 6 - \frac{(p-3)(p-4)}{2}. \qquad (*)$$

It is easy to see that for any positive integer number p,

$$\frac{(p-3)(p-4)}{2} \geqslant 0.$$

So that

$$m \leqslant 3n - 6.$$

Suppose it is true when $t = T - 1$. Let $G_1, G_2, \ldots, G_T$ be the simplical summands of G and $G_2, \ldots, G_T$ be the simplical summands of G'. We may assume that the graph G' has n' vertices and m' edges. By the induction hypothesis, we have

$$m' \leqslant 3n' - 6.$$

Since $G \equiv G_1 \oplus G'$, and $m_1 \leqslant 3n_1 - 6$. The result follows for $t = 2$. Hence

$$m \leqslant 3n - 6. \qquad \square$$

Theorem 3.2. *Let G be a K_5* minor-free *simple graph and with $n \geqslant 5$ vertices, then*

$$\rho(G) \leqslant 1 + \sqrt{3n - 8},$$

where equality holds if and only if G is isomorphic to the graph $K_3 \nabla (n-3)K_1$.

Proof. This follows from Proposition 3.1 and Lemma 2.2. $\square$

Lemma 3.2. *Let G be a simple planar bipartite graph with $n \geqslant 3$ vertices and m edges, then*

$$m \leqslant 2n - 4.$$

Proof. See [5]. $\square$

Proposition 3.2. *Let G be a simple connected bipartite graph with $n \geqslant 5$ vertices and m edges. If G is K_5* minor-free *graph, then*

$$m \leqslant 3n - 9.$$

Proof. Let H be a simple connected edge-maximal *K_5–minor-free* graph with n vertices and m' edges. Suppose that the graphs $H_1, H_2, \ldots, H_t$ are the simplical summands of H and the graph H_i has n_i vertices and m'_i edges. Further, without loss generality, we may assume that G is a spanning subgraph of H. Let the graph G_i be the intersection of G and H_i and with n_i vertices and m_i edges ($1 \leqslant i \leqslant t$).

If there exists a positive integer i such that H_i is isomorphic to the graph V_8, then

$$m' \leqslant 3n - 9.$$

Therefore it follows that

$$m \leqslant 3n - 9.$$

We may assume that H_i is an edge-maximal planar graph for $1 \leqslant i \leqslant t$. Notice that the *p-summing* two edge-maximal planar graphs with at most four vertices is a planar graph. Without loss of generality, we may assume that H_1 is an edge-maximal planar graph with $n_1 \geqslant 5$ vertices and $3n_1 - 6$ edges. Since G_1 is a planar bipartite graph, we have $m_1 \leqslant 2n_1 - 4$. Hence

$$m_1' - m_1 \geqslant n_1 - 2 \geqslant 3.$$

Let $H' \equiv \oplus_{i=2}^{t} H_i$, then $H \equiv H_1 \oplus H'$. For $e \in E(H_1)/E(G_1)$, it is easy to show that $e \notin E(G)$. This implies that

$$m' - m \geqslant 3.$$

By Proposition 3.1, we have

$$m' \leqslant 3n - 6.$$

Therefore

$$m \leqslant 3n - 9. \qquad \square$$

Theorem 3.3. *Let G be a simple connected bipartite graph with $n \geqslant 5$ vertices and $\lambda(G)$ be the smallest eigenvalue of G. If G is K_5*–minor-free, *then*

$$\lambda(G) \geqslant -\sqrt{3n-9},$$

where equality holds if and only if G is isomorphic to the complete bipartite graph $K_{3,n-3}$.

Proof. This follows from Lemmas 2.4, 2.5 and Proposition 3.3. $\square$

Acknowledgements

The author is grateful to the referees for many helpful suggestions which led to an improved version of this paper.

References

[1] S. Arnborg, A. Proskurowski, Characterization and recognition of partial 3-tree, SIAM J. Algebraic Discrete Methods 7 (1986) 305–314.
[2] J.A. Bondy, U.S.R. Murty, Graph Theory with Applications, Macmillan, New York, 1976.
[3] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs, 3rd Edition, Barth, Heidelberg, 1995.
[4] G. Ding, B. Oporowski, D.P. Sanders, Surfaces, tree-width, clique-minors, and partitions, J. Combin. Theory B 79 (2000) 221–246.
[5] Y. Hong, J.L. Shu, Sharp lower bounds of the least eigenvalue of planar graphs, Linear Algebra Appl. 296 (1999) 227–232.
[6] Y. Hong, J.L. Shu, K. Fang, A sharp upper bound of the spectral radius of graphs, J. Combin. Theory B 81 (2001) 177–183.
[7] C. Thomassen, Embeddings and minors, in: R. Graham, M. Grotschel, L. Lovasz (Eds.), Handbook of Combinatorics, Elsevier Science B.V., Amsterdam, 1995, pp. 301–349.
[8] K. Wagner, Uber eine Eigenschaft der ebenen Komplexe, Math. Ann. 114 (1937) 570–590.